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Conjugate gradient method for determining unknown contact conductance during metal casting

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Abstract—The air-gap formation between the casting and the metal mold during the casting of metals creates a thermal resistance that reduces the heat transfer and solidification rates. In this work, the inverse solution methodology based on the *conjugate gradient method* is developed for estimating the variation of air-gap resistance with time from the transient temperature measurements taken with thermocouples inside the casting region and at the outer mold surface. The advantage of the conjugate gradient method is that there is no need to assume a specific functional form for the unknown quantity beforehand, since the solution automatically determines the functional form over the domain specified. Furthermore the method is stable and converges over an order of magnitude faster than the least square method.

1. INTRODUCTION

SEVERAL early studies of casting of metals in metal molds recognized the existence of an interface resistance to heat flow at the mold–casting interface, and assumed the presence of a constant interface resistance throughout the solidification process. However, later studies revealed considerable variation of the interface resistance with time as solidification progressed. A discussion of the problems of air-gap formation at the mold–casting interface and some experimental results on the interface resistance can be found in refs. [1–5]. An inverse analysis of heat conduction involving phase change is essential for accurate determination of air-gap resistance from the transient temperature measurements taken inside the casting region and at the outer mold surface.

Some work has been reported on the inverse analysis of solidification [6–10]. In a recent work [10], the least squares approach is used together with the Levenberg–Marquardt method to determine the contact conductance at the mold–casting interface. Such an approach required a reasonably close first estimate of contact conductance for the solution to converge, even though only four parameters were estimated. In this work, we present the *conjugate gradient method* which converges very rapidly and is not sensitive to the measurement errors.

The conjugate gradient method, by utilizing the ideas based on the variational principles [11, 12], transforms the inverse problems to the solution of three simple problems called the *direct problem*, the *sensitivity problem* and the *adjoint problem* together with the *gradient equation.* In Section 2, the mathematical formulation of the inverse problem is given; Sections 3 and 4 deal, respectively, with the solution of the sensitivity problem for the functions $\Delta T_p(x, t)$ and $\Delta T_c(x, t)$ and the adjoint problem for the functions $\lambda_1(x, t)$ and $\lambda_2(x, t)$. In Section 4 the conjugate gradient method is applied to determine the timewise variation of the unknown contact conductance $h_c(t)$ at the mold–casting interface.

2. PROBLEM FORMULATION

The liquid region is initially at a uniform saturated temperature, T_m. For time $t > 0$ the solidification takes place as a result of convective cooling applied at the mold surface and the solid–liquid interface moves in the positive x direction. Figure 1 shows the geometry and coordinates.

Assuming constant properties, the mathematical formulation of this one-dimensional solidification problem is given by:

Mold region

$$k_p \frac{\partial^2 T_p(x,t)}{\partial x^2} = C_p \frac{\partial T_p(x,t)}{\partial t} \quad \text{in} \quad 0 < x < b, t > 0 \tag{1a}$$

$$-k_p \frac{\partial T_p(0,t)}{\partial x} = h_\infty (T_\infty - T_p) \quad \text{at} \quad x = 0, t > 0 \tag{1b}$$

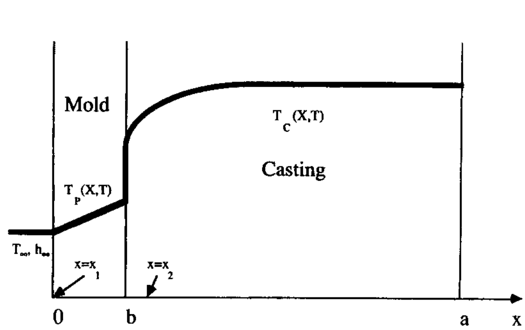

FIG. 1. Geometry and coordinates.

NOMENCLATURE

- $h_c(t)$ unknown contact conductance at the mold–casting interface
- $J(t)$ functional defined by equation (3)
- $J'(t)$ gradient of the functional defined by equation (10)
- k thermal conductivity
- P^k direction of descent at the kth iteration
- $T(x,t)$ estimated temperature
- $\Delta T(x,t)$ sensitivity function satisfying the sensitivity problem defined by equations (4) and (5)
- $Y(x,t)$ measured temperature.

Greek symbols

- α thermal diffusivity
- β^k step size in going from h_c^k to h_c^{k+1}, in equation (11)
- γ^k conjugate coefficient, defined by equation (13)
- δ Dirac delta function
- ε convergence criteria
- $\lambda(x,t)$ adjoint function satisfying the adjoint problem defined by equations (8) and (9)
- σ standard deviation of temperature measurement
- ω random number.

Subscripts

- c casting region
- p mold region.

$$k_p \frac{\partial T_p(b,t)}{\partial x} = h_c(t)(T_c - T_p) \quad \text{at} \quad x = b, t > 0 \qquad (1c)$$

$$T_p(x,0) = T_\infty \quad \text{for} \quad t = 0, \quad \text{in} \quad 0 \leqslant x \leqslant b \qquad (1d)$$

where $C_p = \rho_p c_p$ is the heat capacity per unit volume, ρ_p and c_p are the density and the specific heat of the plate (i.e. mold), respectively.

Casting region

To alleviate the tracking of the moving interface, the enthalpy form of the energy equation is used for the casting region

$$k_c \frac{\partial^2 T_c(x,t)}{\partial x^2} = \rho_c \frac{\partial H_c(x,t)}{\partial t} \quad \text{in} \quad b < x < a, t > 0 \qquad (2a)$$

$$-k_c \frac{\partial T_c(b,t)}{\partial x} = h_c(t)(T_p - T_c) \quad \text{at} \quad x = b, t > 0 \qquad (2b)$$

$$\frac{\partial T_c(a,t)}{\partial x} = 0 \quad \text{at} \quad x = a, t > 0 \qquad (2c)$$

$$T_c(x,0) = T_m \quad \text{for} \quad t = 0, \quad \text{in} \quad b \leqslant x \leqslant a \qquad (2d)$$

where $dH_c = c_c \, dT_c$ is the enthalpy of the casting material, c_c and ρ_c are the specific heat and the density of the casting region, respectively, while T_m is the temperature of the saturated liquid.

The inverse analysis utilizing the conjugate gradient method requires the solution of direct, sensitivity and adjoint problems together with the gradient equation. The direct phase-change problem could be solved by using the standard enthalpy method [13]. The development of sensitivity and adjoint equations and their solutions are discussed next.

3. THE SENSITIVITY PROBLEM

The solution of the problems (1) and (2) with contact conductance $h_c(t)$ unknown, can be recast as a problem of optimum control, i.e. choose the control function $h_c(t)$ such that it minimizes the following functional

$$J(h_c(t)) = \int_{t=0}^{t_f} [(T_1 - Y_1)^2 + (T_2 - Y_2)^2] \, dt \qquad (3)$$

where T_1 and Y_1 are the estimated and measured temperatures, respectively, at the outer mold surface (i.e. $x = x_1$) as shown in Fig. 1. Similarly, T_2 and Y_2 are the estimated and measured temperatures, respectively, in the casting region at a distance Δx_p away from the interface (i.e. $x = x_2$). If an estimate is available for $h_c(t)$, the temperatures T_1 and T_2 can be computed from the solution of the direct problem defined by equations (1) and (2).

It is assumed that when $h_c(t)$ undergoes an increment $\Delta h_c(t)$, then the temperatures $T_p(x,t)$, $T_c(x,t)$ and enthalpy $H_c(x,t)$ change by an amount ΔT_p, ΔT_c and ΔH_c, respectively. To construct the sensitivity problem satisfying the functions ΔT_p, ΔT_c and ΔH_c, we replace T_p by $T_p + \Delta T_p$, T_c by $T_c + \Delta T_c$, H_c by $H_c + \Delta H_c$ and h_c by $h_c + \Delta h_c$ in the direct problems (1) and (2) and then subtract from it the original problems (1) and (2). The following sensitivity problem is obtained for the determination of the functions ΔT_p and ΔT_c in the mold and casting regions, respectively.

Mold region

$$k_p \frac{\partial^2 \Delta T_p(x,t)}{\partial x^2} = C_p \frac{\partial \Delta T_p(x,t)}{\partial t} \quad \text{in} \quad 0 < x < b, t > 0 \qquad (4a)$$

$$-k_p\frac{\partial \Delta T_p(0,t)}{\partial x} = -h_\infty \Delta T_p(0,t) \quad \text{at} \quad x=0, t>0 \tag{4b}$$

$$k_p\frac{\partial \Delta T_p(b,t)}{\partial x} = h_c(t)(\Delta T_c - \Delta T_p) + \Delta h_c(t)(T_c - T_p) \quad \text{at} \quad x=b, t>0 \tag{4c}$$

$$\Delta T_p(x,0) = 0 \quad \text{for} \quad t=0, 0 \leqslant x \leqslant b. \tag{4d}$$

Casting region

$$k_c\frac{\partial^2 \Delta T_c(x,t)}{\partial x^2} = \rho_c\frac{\partial \Delta H_c(x,t)}{\partial t} = C_c\frac{\partial \Delta T_c(x,t)}{\partial t} \quad \text{in} \quad b<x<a, t>0 \tag{5a}$$

$$-k_c\frac{\partial \Delta T_c(b,t)}{\partial x} = h_c(t)(\Delta T_p - \Delta T_c) + \Delta h_c(t)(T_p - T_c) \quad \text{at} \quad x=b, t>0 \tag{5b}$$

$$\frac{\partial \Delta T_c(a,t)}{\partial x} = 0 \quad \text{at} \quad x=a, t>0 \tag{5c}$$

$$\Delta T_c(x,0) = 0 \quad \text{or} \quad \Delta H_c(x,0) = 0 \quad \text{for} \quad t=0, b \leqslant x \leqslant a. \tag{5d}$$

Note that, in equation (5a) we replaced ΔH_c with its equivalent $c_c \Delta T_c$, since this is not a phase change problem; therefore equations (4) and (5) can be solved with the standard finite difference techniques.

4. THE ADJOINT PROBLEM

To derive the adjoint equation we multiply equations (1a) and (2a) with the adjoint functions $\lambda_1(x, t)$ and $\lambda_2(x, t)$, respectively, integrate the resulting expression over the total time t_f and the total space domain $0 < x < b$ and $b < x < a$ and then add this result to the functional given by equation (3). The following expression results:

$$J(t) = \int_{t=0}^{t_f} [(T_1 - Y_1)^2 + (T_2 - Y_2)^2]\,dt + \int_{t=0}^{t_f}\int_{x=0}^{b} \lambda_1(x,t)\left[k_p\frac{\partial^2 T_p}{\partial x^2} - C_p\frac{\partial T_p}{\partial t}\right]dx\,dt + \int_{t=0}^{t_f}\int_{x=b}^{a} \lambda_2(x,t)\left[k_c\frac{\partial^2 T_c}{\partial x^2} - \rho_c\frac{\partial H_c}{\partial t}\right]dx\,dt. \tag{6}$$

The variation $\Delta J(t)$ of equation (6) is then obtained by the variation principle [11, 12] as follows:

$$\Delta J(t) = \int_{t=0}^{t_f}\int_{x=0}^{b} 2(T_1 - Y_1)\Delta T_p(x,t)\delta(x - x_1)\,dx\,dt + \int_{t=0}^{t_f}\int_{x=b}^{a} 2(T_2 - Y_2)\Delta T_c(x,t)\delta(x - x_2)\,dx\,dt + \int_{t=0}^{t_f}\int_{x=0}^{b} \lambda_1(x,t)\left[k_p\frac{\partial^2 \Delta T_p}{\partial x^2} - C_p\frac{\partial \Delta T_p}{\partial t}\right]dx\,dt + \int_{t=0}^{t_f}\int_{x=b}^{a} \lambda_2(x,t)\left[k_c\frac{\partial^2 \Delta T_c}{\partial x^2} - \rho_c\frac{\partial \Delta H_c}{\partial t}\right]dx\,dt \tag{7}$$

where $\delta(x)$ is the Dirac delta function, x_1 and x_2 are the locations of thermocouples. The last two integral terms in equation (7) are integrated by parts and the boundary conditions of the sensitivity problem defined by equations (4) and (5) are utilized to obtain the following coupled *adjoint problem* given by equations (8) and (9) and the *gradient equation* (10).

Mold region

$$k_p\frac{\partial^2 \lambda_1(x,t)}{\partial x^2} + C_p\frac{\partial \lambda_1(x,t)}{\partial t} + 2(T_1 - Y_1) = 0 \quad \text{in} \quad 0<x<b, t>0 \tag{8a}$$

$$-k_p\frac{\partial \lambda_1(0,t)}{\partial x} = -h_\infty \lambda_1(0,t) \quad \text{at} \quad x=0, t>0 \tag{8b}$$

$$k_p\frac{\partial \lambda_1(b,t)}{\partial x} = h_c(t)(\lambda_2 - \lambda_1) \quad \text{at} \quad x=b, t>0 \tag{8c}$$

$$\lambda_1(x,t_f) = 0 \quad \text{for} \quad t=t_f, \quad \text{in} \quad 0 \leqslant x \leqslant b. \tag{8d}$$

Casting region

$$k_c\frac{\partial^2 \lambda_2(x,t)}{\partial x^2} + C_c\frac{\partial \lambda_2(x,t)}{\partial t} + 2(T_2 - Y_2) = 0 \quad \text{in} \quad b<x<a, t>0 \tag{9a}$$

$$-k_c\frac{\partial \lambda_2(b,t)}{\partial x} = h_c(t)(\lambda_1 - \lambda_2) \quad \text{at} \quad x=b, t>0 \tag{9b}$$

$$\frac{\partial \lambda_2(a,t)}{\partial x} = 0 \quad \text{at} \quad x=a, t>0 \tag{9c}$$

$$\lambda_2(x,t_f) = 0 \quad \text{for} \quad t=t_f, \quad \text{in} \quad b \leqslant x \leqslant a. \tag{9d}$$

Gradient equation

The gradient equation for the functional, $J(t)$, is given in the form

$$J'(t) = [\lambda_2(b,t) - \lambda_1(b,t)][T_p(b,t) - T_c(b,t)]. \tag{10}$$

Note that the problems (8) and (9) are not the phase change problem, therefore they can be solved with the standard finite difference techniques.

5. INVERSE SOLUTION BY CONJUGATE GRADIENT METHOD

In this section an algorithm is presented for solving the inverse heat conduction problem described previously with the conjugate gradient method. The method is stable and converges very fast if some information is available for the final time condition of the unknown function $h_c(t_f)$. In this study, an estimate is made for the final time condition of $h_c(t_f)$ in the following manner.

The mold region being very thin, we assume a linear variation of temperature (i.e. constant temperature gradient) within the mold for each time step and compute the gradient $\partial T_p/\partial x$ using equation (1b) and the measured data Y_1 at $x = x_1$. Knowing this gradient, the interface temperature is determined from equations (1c), (2b) and the measured data Y_2 at $x = x_2$ is considered available. One can use either of the equations (1c) or (2b) to determine a first estimate of the final time condition of the contact conductance $h_c(t_f) = h_c^0(t)$.

The following iterative procedure [11] is used for the determination of the contact conductance

$$h_c^{k+1} = h_c^k - \beta^k P^k; \quad k = 0, 1, 2, \ldots \tag{11}$$

where the direction of descent P^k is determined from the following relation

$$P^k = J'^k + \gamma^k P^{k-1} \tag{12}$$

here P^{k-1} is the value of P at step $k-1$ and J'^k is the value of the gradient of the functional at step k.

Different definition of the conjugate coefficient γ^k can be found in the standard texts on mathematics, we choose the form [14, 15]

$$\gamma^k = \frac{\int_{t=0}^{t_f} [J'^k(t)]^2 \, dt}{\int_{t=0}^{t_f} [J'^{k-1}(t)]^2 \, dt} \quad \text{with} \quad \gamma^0 = 0. \tag{13}$$

The coefficient β^k, which determines the *step size* in going from h_c^k to h_c^{k+1} in equation (11) is obtained by minimizing $J(h_c^{k+1})$ with respect to β^k, i.e.

$$\min_\beta J(h_c^{k+1}) = \min_\beta \int_{t=0}^{t_f} \{[T_1(h_c^k - \beta^k P^k) - Y_1]^2 + [T_2(h_c^k - \beta^k P^k) - Y_2]^2\} \, dt. \tag{14a}$$

First, the Taylor series expansion is used to linearize the right-hand side of this expression in the form

$$\min_\beta J(h_c^{k+1}) = \min_\beta \int_{t=0}^{t_f} \{[T_1(h_c^k) - \beta^k \Delta T_1(P^k) - Y_1]^2 + [T_2(h_c^k) - \beta^k \Delta T_2(P^k) - Y_2]^2\} \, dt. \tag{14b}$$

Then equation (14b) is minimized by differentiating it with respect to β^k and equating it equal to zero. After rearrangement, the following expression is obtained for step size β^k

$$\frac{\int_{t=0}^{t_f} \{\Delta T_1(P^k)[T_1(h_c^k) - Y_1] + \Delta T_2(P^k)[T_2(h_c^k) - Y_2]\} \, dt}{\int_{t=0}^{t_f} [\Delta T_1^2(P^k) + \Delta T_2^2(P^k)] \, dt}. \tag{15}$$

The final time condition $h_c(t_f)$ determined as described above, i.e. $h_c^0(t) = h_c(t_f)$ is used to start the iterations.

Once P^k is computed from equation (12) and β^k from equation (15), the iterative process defined by equation (11) can be applied to determine h_c^{k+1} until a specified stopping criterion based on the *discrepancy principle* described below is satisfied.

Discrepancy principle for stopping criteria

If the problem involves no measurement errors, the traditional check condition specified as [16]

$$J(h_c^{k+1}) < \varepsilon_1 \tag{16}$$

where ε_1 is a small specified number, could be used. However, the observed temperature data contain measurement errors; as a result, the inverse solution will tend to approach the perturbed input data and the solution will exhibit oscillatory behavior as the number of iterations is increased [11]. Computational experience shows that it is advisable to use the discrepancy principle [17, 18] for terminating the iteration process. The discrepancy principle that establishes the value of ε from equation (3) by assuming $(T_1 - Y_1) \cong (T_2 - Y_2) \cong \sigma$, is given in the form

$$2 \int_{t=0}^{t_f} \sigma^2 \, dt \equiv \varepsilon^2 \tag{17a}$$

where σ is the standard deviation of the measurement error. This value of ε is then used as the stopping criterion, i.e.

$$J(h_c^{k+1}) < \varepsilon^2. \tag{17b}$$

6. THE ALGORITHM

The algorithm for the computational procedure of the iterative scheme starting from the kth iteration is summarized as:

STEP (1) $h_c^k(t)$ is available at the kth iteration. Solve the direct problem given by equations (1) and (2) and compute $T_p(x, t)$ and $T_c(x, t)$.

STEP (2) Knowing $T_p(x, t)$, $T_c(x, t)$ and measured temperatures Y_1, Y_2, solve the adjoint problem defined by equations (8), (9) and obtain the adjoint variables $\lambda_1(x, t)$ and $\lambda_2(x, t)$.

STEP (3) Knowing $\lambda_1(b, t)$, $\lambda_2(b, t)$ and $T_p(b, t)$, $T_c(b, t)$, compute the gradient of the functional, $J'(t)$, from equation (10).

STEP (4) Knowing $J'(t)$, first compute γ_k from equation (13) and then compute the direction of descent P^k from equation (12).

STEP (5) Knowing the direction of descent P^k, solve the sensitivity problem given by equations (4), (5) and determine the sensitivity functions $\Delta T_1(P^k)$ and $\Delta T_2(P^k)$.

STEP (6) Knowing $\Delta T_1(P^k)$ and $\Delta T_2(P^k)$, compute step size β^k from equation (15).

STEP (7) Knowing step size β^k, compute new contact conductance $h_c^{k+1}(t)$ from equation (11).

STEP (8) Check if the stopping criterion given by equation (17) satisfied.

STEP (9) If not, repeat the above calculational procedure until the stopping criterion given by equation (17) is satisfied.

7. RESULTS AND DISCUSSIONS

To illustrate the accuracy of the present approach in predicting $h_c(t)$ with inverse analysis, we first examine two very strict test cases involving a triangular and a step contact conductance function and then examine the case studied in ref. [10].

Here we consider the mold and casting are initially at constant temperatures T_∞ and T_m, respectively, and the thermal properties of the liquid and solid phases are constant and equal. The following physical quantities are used in the calculation [10]:

$$k_p = 388\ \mathrm{W\,m^{-1}\,{}^\circ C^{-1}} \quad k_c = 213\ \mathrm{W\,m^{-1}\,{}^\circ C^{-1}}$$

$$c_p = 403\ \mathrm{J\,kg^{-1}\,{}^\circ C^{-1}} \quad c_c = 1210\ \mathrm{J\,kg^{-1}\,{}^\circ C^{-1}}$$

$$\rho_p = 8940\ \mathrm{kg\,m^{-3}} \quad \rho_c = 2700\ \mathrm{kg\,m^{-3}}$$

$$h_\infty = 2000\,\mathrm{W\,m^{-2}\,{}^\circ C^{-1}} \quad T_\infty = 20^\circ\mathrm{C} \quad b = 6\times10^{-3}\,\mathrm{m}$$

$$a = 0.206\,\mathrm{m} \quad L = 4.02\times10^5\,\mathrm{J\,kg^{-1}} \quad T_m = 660^\circ\mathrm{C}.$$

The total measurement is taken over a period of 80 s with the measurement time step, $\Delta t = 1$ s. Thus 80 temperature readings per thermocouple over the total measurement time. The space steps for mold region, Δx_p, and for casting region, Δx_c, are taken as 3×10^{-3} m and 1.6×10^{-3} m, respectively. One thermocouple is placed at the outer surface of the mold and the other is located inside the casting region at a location 1.6×10^{-3} m from the interface.

The measured temperature data, Y, are generated by adding a standard deviation σ to the simulated exact temperature, given by

$$Y_{\mathrm{measured}} = Y_{\mathrm{exact}} + \omega\sigma \tag{18}$$

where the random variable ω is calculated by the IMSL subroutine DRNNOR [19]. In the present calculation the range of ω is chosen as $-2.576 < \omega < 2.576$ which represents the 99% confidence bound for the measurement temperature.

We present below three numerical experiments in predicting the timewise variation of $h_c(t)$ by inverse analysis

Case 1. Triangular jump in $h_c(t)$.

The interface contact conductance $h_c(t)$ is assumed to vary in the form

$$h_c(t) = \begin{cases} 300 + 15t & 0 \leqslant t \leqslant 20 \\ 200 - \dfrac{80}{7}(t-55) & 20 \leqslant t \leqslant 55 \\ 200 & 55 \leqslant t \leqslant 80 \end{cases} \tag{19}$$

which represents a triangular function with the jump, $h_c(0) = 300$, at $t = 0$.

The inverse solutions for $\sigma = 0.0$ (exact), $\sigma = 1.0$ and $\sigma = 2.0$ are obtained by the conjugate gradient approach as shown in Figs. 2, 3 and 4, respectively. With no measurement error, the estimated contact conductance is very close to the exact value as given

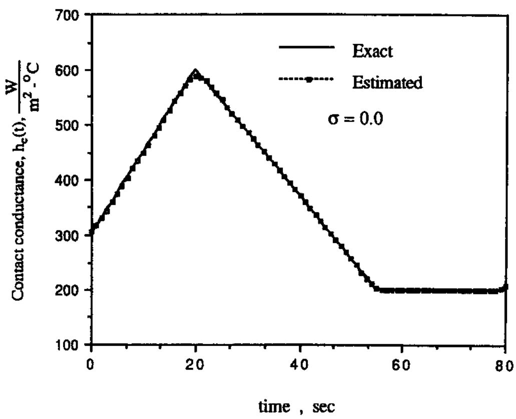

FIG. 2. Estimated contact conductance for case 1 with no measurement errors.

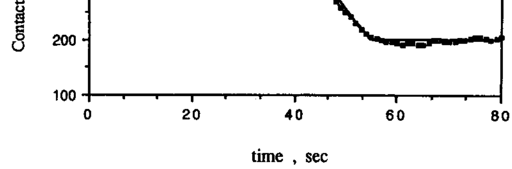

FIG. 3. Estimated contact conductance for case 1 with measurement errors $\sigma = 1.0$.

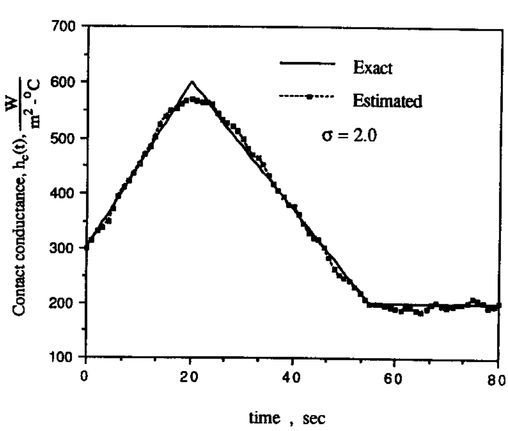

FIG. 4. Estimated contact conductance for case 1 with measurement errors $\sigma = 2.0$.

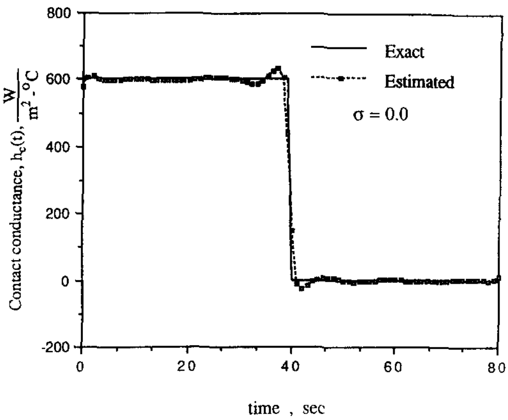

FIG. 5. Estimated contact conductance for case 2 with no measurement errors.

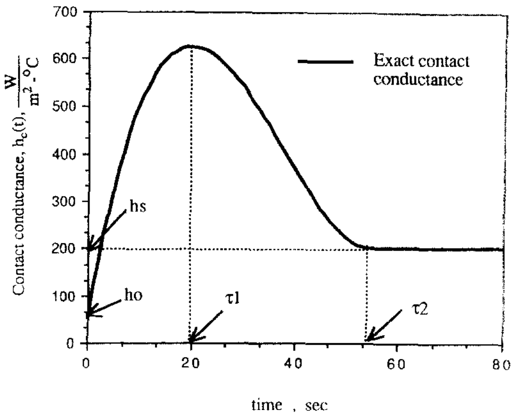

FIG. 7. Variation of $h_c(t)$ with time.

in Fig. 2. As measurement error is included, the accuracy of the inverse solution decreases, as shown in Figs. 3 and 4. We note that, even with $\sigma = 2.0$ the results are still good as shown in Fig. 4.

Case 2. Step function variation in $h_c(t)$.

The interface contact conductance $h_c(t)$ is assumed in the form

$$h_c(t) = \begin{cases} 600 & 0 \leqslant t \leqslant 40 \\ 0 & 40 < t \leqslant 80 \end{cases} \tag{20}$$

which is a step jump function with $h_c(0) = 600$ at $t = 0$ and represents a very strict test for the accuracy of the prediction.

The inverse solutions for $\sigma = 0.0$ (exact) and $\sigma = 1.0$ are shown in Figs. 5 and 6, respectively. For the case $\sigma = 0.0$, slight deviation of the results occurs at the sharp corners only, but the agreement is very good for the rest of the function. Results are still good for the case $\sigma = 1.0$.

Case 3. A polynomial variation in $h_c(t)$.

To compare the conjugate gradient method with the least square method utilizing the Levenberg–Marquardt algorithm, the interface conductance $h_c(t)$ is assumed to be a cubic polynomial in time, given in the form [10]

$$h_c(t) = A_1 + A_2 t + A_3 t^2 + A_4 t^3 \quad \text{for} \quad t < \tau_2 \tag{21a}$$

and

$$h_c(t) = h_s \quad \text{for} \quad t \geqslant \tau_2 \tag{21b}$$

where the coefficients A_i, $i = 1, 4$ are established by the following requirements based on some physical considerations

$$h_c(t) = h_0 \quad \text{at} \quad t = 0 \tag{22a}$$

$$\frac{\partial h_c(t)}{\partial t} = 0 \quad \text{at} \quad t = \tau_1 \tag{22b}$$

$$h_c(t) = h_s \quad \text{at} \quad t = \tau_2 \tag{22c}$$

$$\frac{\partial h_c(t)}{\partial t} = 0 \quad \text{at} \quad t = \tau_2. \tag{22d}$$

The physical significance of the four parameters h_0, h_s, τ_1 and τ_2 is illustrated in Fig. 7. The values of these four parameters characterizing the interface conductance $h_c(t)$ are taken as $h_0 = 50\ \text{W m}^{-2}\ °\text{C}^{-1}$, $h_s = 200\ \text{W m}^{-2}\ °\text{C}^{-1}$, $\tau_1 = 20$ s and $\tau_2 = 55$ s. Figure 8 shows our prediction of the functional form of $h_c(t)$

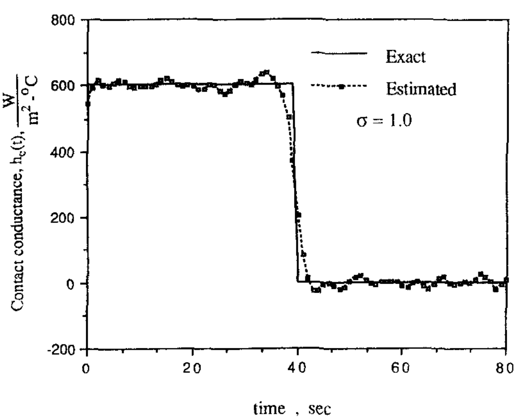

FIG. 6. Estimated contact conductance for case 2 with measurement errors $\sigma = 1.0$.

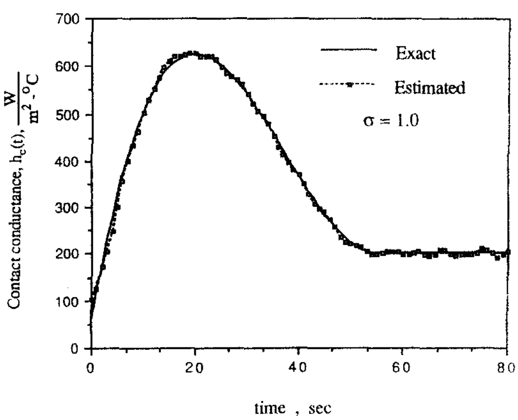

FIG. 8. Estimated contact conductance for case 3 with measurement errors $\sigma = 1.0$.

by the conjugate gradient method for the case $\sigma = 1.0$ and its comparison with the exact values of $h_c(t)$. Clearly, the prediction is in excellent agreement with the exact $h_c(t)$.

The foregoing comparison shows that the function estimation approach utilizing the conjugate gradient method requires less computer time, no a priori assumption in the functional form of the unknown quantity and the method is less sensitive to the measurement errors.

In the function estimation approach considered here a total of 80 unknowns are estimated to establish the unknown function, whereas in the least squares method four parameters were used to represent the function. The computer time requirement with the present approach was an order of magnitude less than that for the least squares method.

8. CONCLUSION

The conjugate gradient method which utilizes the function estimation approach is used to solve the inverse solidification problem to determine the unknown timewise variation of the contact conductance between the mold and casting region. The results show that the conjugate gradient method requires much less computer time than the least squares method, less sensitive to the measurement errors and does not require a prior information for the functional form of the unknown quantity.

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METHODE DES GRADIENTS CONJUGUES POUR DETERMINER LA CONDUCTANCE INCONNUE DE CONTACT PENDANT LE MOULAGE D'UN METAL

Résumé—La formation d'un matelas d'air entre le métal et le moule pendant la coulée crée une résistance thermique qui réduit les transferts thermiques et les vitesses de solidification. On développe la méthodologie de solution inverse basée sur *la méthode des gradients conjugués* pour estimer la variation de la résistance variable du matelas d'air à partir des mesures de température avec des thermocouples dans la région liquide et à la surface du moule externe. L'avantage de cette méthode est de ne pas supposer une forme fonctionnelle de la grandeur inconnue, puisque la solution détermine automatiquement la forme fonctionnelle sur le domaine spécifié. La méthode est stable et elle converge sur un ordre de grandeur, plus vite que la méthode des moindres carrés.

EIN VERFAHREN MIT KONJUGIERTEN GRADIENTEN ZUR BESTIMMUNG DES UNBEKANNTEN KONTAKTWIDERSTANDES BEIM GIESSEN VON METALLEN

Zusammenfassung—Beim Gießen von Metallen entsteht durch die Ausbildung eines Luftspaltes zwischen der Gußform und dem flüssigen Metall ein thermischer Widerstand, der den Wärmeübergang und die Verfestigungsgeschwindigkeit reduziert. In der vorliegenden Arbeit wird auf der Grundlage des Verfahrens der konjugierten Gradienten eine umgekehrte Methode zur Bestimmung der zeitlichen Veränderung des Widerstandes durch den Luftspalt entwickelt. Die Grundlage dafür bildet die Messung der zeitlich veränderten Temperaturen mit Hilfe von Thermoelementen im Gußgebiet und an der äußeren Oberfläche der Gußform. Der Vorteil des Verfahrens mit konjugierten Gradienten besteht darin, daß nicht von vornherein eine spezielle Funktionsform für die unbekannte Größe angenommen werden muß. Diese Form ergibt sich automatisch für den angegebenen Bereich. Darüberhinaus ist das Verfahren stabil und konvergiert um Größenordnungen schneller als die übliche Fehlerquadratmethode.

МЕТОД СОПРЯЖЕННЫХ ГРАДИЕНТОВ ДЛЯ ОПРЕДЕЛЕНИЯ КОНТАКТНОЙ ПРОВОДИМОСТИ В ПРОЦЕССЕ ЛИТЬЯ МЕТАЛЛОВ

Аннотация—Образование воздушных зазоров между отливкой и кокилем в процессе литья металлов создает тепловое сопротивление, снижающее скорости теплопереноса и затвердевания. В настоящем исследовании разработана методика решения обратной задачи на основе метода сопряженных градиентов, позволяющая оценить изменение сопротивления воздушных зазоров со временем исходя из измерений температуры с использованием термопар, помещенных в области отливки и на внешней границе кокиля. Преимущество предложенного метода состоит в отсутствии необходимости предварительного предположения о конкретной функциональной форме неизвестной величины, поскольку решение автоматически определяет ее в заданной области. Кроме того, используемый метод является устойчивым и сходится на порядок быстрее, чем метод наименьших квадратов.